



Differential Geometry II - Smooth Manifolds

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## Exercise Sheet 8 – Solutions

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**Exercise 1:** Let  $M$  be a smooth manifold. Show that if  $S$  is an embedded submanifold of  $M$ , then the subspace topology on  $S$  and the smooth structure on  $S$  described in *Theorem 5.9* are the only topology and smooth structure with respect to which  $S$  is an embedded (or immersed) submanifold.

[Hint: Use [*Exercise Sheet 7, Exercise 5(c)*].]

**Solution:** Consider some other topology and smooth structure on  $S$ , denote by  $\tilde{S}$  the resulting smooth manifold, and suppose that  $\tilde{\iota}: \tilde{S} \hookrightarrow M$  is a smooth immersion. (For the exercise as stated, one can suppose that  $\tilde{\iota}$  is a smooth embedding, but the weaker assumption that it is a smooth immersion is actually sufficient.) By [*Exercise Sheet 7, Exercise 5(c)*] we infer that the corestriction  $\tilde{\iota}|^S: \tilde{S} \rightarrow S$  is smooth as well. If we denote by  $\iota: S \hookrightarrow M$  the inclusion map of  $S$  into  $M$ , then we have  $\iota \circ (\tilde{\iota}|^S) = \tilde{\iota}$ , so given  $p \in \tilde{S}$ , by taking differentials we obtain

$$d\iota_p \circ d(\tilde{\iota}|^S)_p = d\tilde{\iota}_p.$$

Since  $d\iota_p$  and  $d\tilde{\iota}_p$  are injective, we deduce that  $d(\tilde{\iota}|^S)_p$  is injective as well. Hence,  $\tilde{\iota}|^S$  is a smooth immersion, and as it is also bijective, by the *Global Rank Theorem* (see *Theorem 4.14*) we conclude that  $\tilde{\iota}|^S$  is a diffeomorphism. Since it is the identity on the underlying set  $S$ , we deduce that the topology and smooth structure of  $\tilde{S}$  are identical to the ones of  $S$ .

*Remark.* Thanks to this uniqueness result, we now know that a subset  $S \subseteq M$  is an embedded submanifold if and only if it satisfies the local slice condition, and if so, its topology and smooth structure are uniquely determined. Since the local slice condition is a local condition, if every point  $p \in S$  has a neighborhood  $U \subseteq M$  such that  $U \cap S$  is an embedded  $k$ -submanifold of  $U$ , then  $S$  is an embedded  $k$ -submanifold of  $M$ .

**Exercise 2:** Let  $M$  be a smooth manifold. Show that if  $S$  is an immersed submanifold of  $M$ , then for the given topology on  $S$ , there exists only one smooth structure making  $S$  into an immersed submanifold.

[Hint: Use [*Exercise Sheet 7, Exercise 5(b)*].]

**Solution:** Denote by  $\iota$  the inclusion map  $S \hookrightarrow M$  of the immersed submanifold  $S$  of  $M$  and by  $\tilde{S}$  the topological space  $S$  endowed now with another smooth structure such that the inclusion map  $\tilde{\iota}: \tilde{S} \hookrightarrow M$  is a smooth immersion. Note that  $\tilde{S}$  is an immersed submanifold of  $M$ . Since  $S$  and  $\tilde{S}$  have the same topology by assumption, both maps  $\iota: S \rightarrow \tilde{S}$  and  $\tilde{\iota}: \tilde{S} \rightarrow S$  are continuous, hence smooth by [Exercise Sheet 7, Exercise 5(b)], so they are inverses of each other. Therefore,  $S$  is diffeomorphic to  $\tilde{S}$ .

*Remark.* It is certainly possible for a given subset  $S$  of a smooth manifold  $M$  to have more than one topology making it into an immersed submanifold of  $M$ . However, for *weakly embedded submanifolds* we have the following uniqueness result, which can be proved similarly to Exercise 2: *If  $M$  is a smooth manifold and if  $S$  is a weakly embedded submanifold of  $M$ , then  $S$  has only one topology and smooth structure with respect to which it is an immersed submanifold of  $M$ .*

**Exercise 3:**

- (a) Let  $M$  be a smooth manifold, let  $S \subseteq M$  be an immersed or embedded submanifold, and let  $p \in S$ . Show that a vector  $v \in T_pM$  is in  $T_pS$  if and only if there exists a smooth curve  $\gamma: J \rightarrow M$  whose image is contained in  $S$ , and which is also smooth as a map into  $S$ , such that  $0 \in J$ ,  $\gamma(0) = p$  and  $\gamma'(0) = v$ .
- (b) Let  $M$  be a smooth manifold, let  $S \subseteq M$  be an embedded submanifold and let  $\gamma: J \rightarrow M$  be a smooth curve whose image happens to lie in  $S$ . Show that  $\gamma'(t)$  is in the subspace  $T_{\gamma(t)}S$  of  $T_{\gamma(t)}M$ .
- (c) Give a counterexample to part (b) if  $S \subseteq M$  is immersed but not embedded.  
[Hint: Example 4.5(2).]

**Solution:**

(a) Assume that the given vector  $v \in T_pM$  lies also in  $T_pS$ , which is identified with  $d_{\iota_p}(T_pS)$ , so that  $v = d_{\iota_p}(w)$  for some  $w \in T_pS$ . By [Exercise Sheet 4, Exercise 5(a)] there exists a smooth curve  $\gamma: J \rightarrow S$  such that  $0 \in J$ ,  $\gamma(0) = p$  and  $\gamma'(0) = w$ . Since  $S$  is an immersed (or embedded) submanifold of  $M$ , the inclusion map  $\iota: S \hookrightarrow M$  is a smooth immersion, so the composite map  $\iota \circ \gamma: J \rightarrow M$  is a smooth curve in  $M$  whose image is clearly contained in  $S$ , it satisfies  $0 \in J$ ,  $(\iota \circ \gamma)(0) = p$ , and additionally by [Exercise Sheet 4, Exercise 5(b)] we have

$$(\iota \circ \gamma)'(0) = d_{\iota_{\gamma(0)}}(\gamma'(0)) = d_{\iota_p}(w) = v.$$

The converse follows immediately from [Exercise Sheet 4, Exercise 5(a)] taking the identification of  $T_pS$  with  $d_{\iota_p}(T_pS)$  into account.

(b) By assumption and by [Exercise Sheet 7, Exercise 5(c)] the given map  $\gamma: J \rightarrow M$  is also smooth as a map from  $J$  to  $S$ , so the statement follows immediately from part (a).

(c) Consider the smooth map

$$\beta: (-\pi, \pi) \rightarrow \mathbb{R}^2, t \mapsto (\sin 2t, \sin t)$$

whose image  $S := \beta(-\pi, \pi)$  is the figure-eight curve in  $\mathbb{R}^2$ . According to *Proposition 5.16* and [*Exercise Sheet 7, Exercise 3*],  $S$  is an immersed but not an embedded submanifold of  $\mathbb{R}^2$ . Observe that  $\beta(0) = (0, 0)$  and that

$$\begin{aligned} d\beta_0 \left( \left. \frac{d}{dt} \right|_{t=0} \right) &= 2 \cos(0) \left. \frac{\partial}{\partial x} \right|_{(0,0)} + \cos(0) \left. \frac{\partial}{\partial y} \right|_{(0,0)} \\ &= 2 \left. \frac{\partial}{\partial x} \right|_{(0,0)} + \left. \frac{\partial}{\partial y} \right|_{(0,0)}. \end{aligned}$$

Since  $\beta: (-\pi, \pi) \rightarrow S$  is a diffeomorphism by *Proposition 5.16*, the tangent vector  $2 \left. \frac{\partial}{\partial x} \right|_{(0,0)} + \left. \frac{\partial}{\partial y} \right|_{(0,0)}$  constitutes a basis for  $T_{(0,0)}S$ .

Consider now the smooth map

$$\gamma: (-\pi, \pi) \rightarrow \mathbb{R}^2, \quad t \mapsto (\sin 2t, -\sin t)$$

and observe that its image lies in  $S$ . Moreover,  $\gamma(0) = (0, 0)$  and we have

$$d\gamma_0 \left( \left. \frac{d}{dt} \right|_{t=0} \right) = 2 \left. \frac{\partial}{\partial x} \right|_{(0,0)} - \left. \frac{\partial}{\partial y} \right|_{(0,0)},$$

which clearly does not lie in the subspace  $T_{(0,0)}S$  of  $T_{(0,0)}\mathbb{R}^2$ .

*Remark.* The same (counter)example shows that the characterization of  $T_pS$  given in *Proposition 5.20* does not work in the merely immersed case.

#### Exercise 4:

- (a) Let  $M$  be a smooth manifold and let  $S \subseteq M$  be an embedded submanifold. Show that if  $\Phi: U \rightarrow N$  is a local defining map for  $S$ , then it holds that

$$T_pS \cong \ker(d\Phi_p: T_pM \rightarrow T_{\Phi(p)}N) \quad \text{for every } p \in S \cap U.$$

- (b) Let  $M$  be a smooth manifold. Suppose that  $S \subseteq M$  is a level set of a smooth submersion  $\Phi = (\Phi_1, \dots, \Phi_k): M \rightarrow \mathbb{R}^k$ . Show that a vector  $v \in T_pM$  is tangent to  $S$  if and only if  $v\Phi_1 = \dots = v\Phi_k = 0$ .

#### Solution:

- (a) Recall that we identify  $T_pS$  with its image  $d\iota_p(T_pS) \subseteq T_pM$ , where  $\iota: S \hookrightarrow M$  is the inclusion map, which is a smooth embedding by assumption. Note that by hypothesis we have  $S \cap U = \Phi^{-1}(q)$  for some  $q \in N$ . Therefore,

$$\Phi \circ \iota|_{S \cap U} = c_q,$$

where  $c_q: S \cap U \rightarrow N$  is the constant map on  $S \cap U$  with value  $q \in N$ . Thus, if  $p \in S \cap U$  is arbitrary, then

$$0 = d(c_q)_p = d\Phi_p \circ d(\iota|_{S \cap U})_p.$$

Hence, the differential  $d(\iota|_{S \cap U})_p$  induces an injective map from  $T_p S$  to  $\ker d\Phi_p$ , because  $\iota$  is an embedding.

In order to conclude, it suffices to show that both spaces have the same dimension. Denote by  $m, n, s$  the dimension of  $M, N, S$ , respectively. By *Corollary 5.13* the codimension of  $S$  in  $M$  is  $n$  (i.e.,  $m - s = n$ ). On the other hand, by the *rank-nullity theorem* from linear algebra and by the surjectivity of  $d\Phi_p$  we obtain

$$\begin{aligned} n &= \dim(\operatorname{im} d\Phi_p) = \underbrace{\dim T_p M}_{=m} - \dim(\ker d\Phi_p) \\ \implies \dim(\ker d\Phi_p) &= m - n = s. \end{aligned}$$

Hence,  $T_p S$  and  $\ker d\Phi_p$  have the same dimension  $s$ , and are thus identified via  $d\iota_p$ .

(b) Fix  $p \in S$ . By part (a) we know that  $v \in T_p M$  is tangent to  $S$  if and only if  $d\Phi_p(v) = 0$ . Denote by  $\operatorname{pr}_1, \dots, \operatorname{pr}_k: \mathbb{R}^k \rightarrow \mathbb{R}$  the projection maps to the corresponding coordinates. By the description of  $T_p \mathbb{R}^k$ , note that a vector  $w \in T_p \mathbb{R}^k$  is 0 if and only if  $w(\operatorname{pr}_i) = 0$  for all  $1 \leq i \leq k$ . Hence,

$$d\Phi_p(v) = 0 \iff d\Phi_p(v)(\operatorname{pr}_i) = 0, \forall 1 \leq i \leq k \iff v(\operatorname{pr}_i \circ \Phi) = v\Phi_i = 0, \forall 1 \leq i \leq k.$$

**Exercise 5 (to be submitted):**

(a) Consider the smooth function

$$\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 - y^2.$$

Show that the level set  $\Phi^{-1}(0)$  is an immersed submanifold of  $\mathbb{R}^2$ .

[Hint: View  $\Phi^{-1}(0)$  as the disjoint union of two lines (where we remove the origin from one of them).]

(b) Consider the smooth function

$$\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 - y^3.$$

Show that the level set  $\Psi^{-1}(0)$  is not an immersed submanifold of  $\mathbb{R}^2$ .

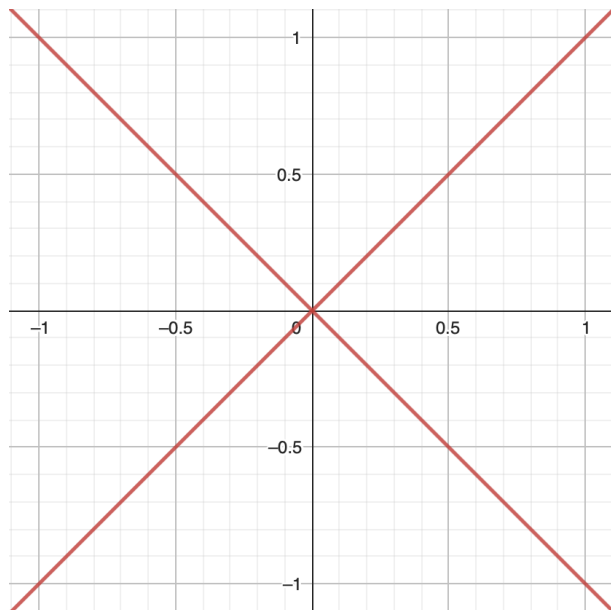
[Hint: Argue by contradiction and use *Exercise 3(a)*.]

**Solution:**

(a) The level set

$$\begin{aligned} \Phi^{-1}(0) &= \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid (y - x)(y + x) = 0\} \end{aligned}$$

is plotted below.



Even though it is not an embedded submanifold of  $\mathbb{R}^2$ , as already demonstrated in the solution of [Exercise Sheet 7, Exercise 4(b)], we will show that  $\Phi^{-1}(0)$  is the image of an injective smooth immersion, and hence  $\Phi^{-1}(0)$  can be given a topology and smooth structure making it into an immersed submanifold of  $\mathbb{R}^2$ ; see Proposition 5.16.

Recall that the problem lies at the point where the two lines cross, which is the origin  $(0,0) \in \mathbb{R}^2$  in our case. The idea is to view  $\Phi^{-1}(0)$  as the disjoint union of two lines, where we remove the origin from one of them. To make this precise, let us start with a general construction for smooth manifolds, namely the *disjoint union*. For  $i \in \{0,1\}$ , let  $(M_i, \mathcal{T}_i, \mathcal{A}_i)$  be a smooth manifold, and assume that both of them have the same dimension. The set-theoretic disjoint union of  $M_0$  and  $M_1$  is the set

$$M_0 \sqcup M_1 := \{(m, i) \mid i \in \{0, 1\}, m \in M_i\}.$$

We can endow the set  $M_0 \sqcup M_1$  with a natural topology  $\mathcal{T}$ , called the *disjoint union topology* as follows. For each  $i \in \{0, 1\}$ , denote by  $\iota_i$  the natural inclusion

$$\begin{aligned} \iota_i: M_i &\rightarrow M_0 \sqcup M_1 \\ m &\mapsto (m, i) \end{aligned}$$

and define  $\mathcal{T}$  by

$$\mathcal{T} := \{U \subseteq M_0 \sqcup M_1 \mid \forall i \in \{0, 1\} : \iota_i^{-1}(U) \in \mathcal{T}_i\}.$$

It is straightforward to check that  $\mathcal{T}$  is a topology on  $M_0 \sqcup M_1$ , and in fact it is the finest topology on  $M_0 \sqcup M_1$  for which the inclusions  $\iota_i$  are continuous. Furthermore, one can observe that  $\iota_i$  is an open map, and as it is injective,  $\iota_i$  is a homeomorphism onto the open subset  $\iota_i(M_i)$  of  $M_0 \sqcup M_1$ . Therefore, we can identify  $(M_i, \mathcal{T}_i)$  with  $\iota_i(M_i)$  endowed with the subspace topology (note also that the open subsets of  $M_0 \sqcup M_1$  are precisely the subsets of the form  $\iota_0(U_0) \cup \iota_1(U_1)$  where  $U_0 \subseteq M_0$  resp.  $U_1 \subseteq M_1$  are open). In particular,  $M_0 \sqcup M_1$  is locally Euclidean and second countable, since we have the open cover  $M_0 \sqcup M_1 = \iota_0(M_0) \cup \iota_1(M_1)$ , where both open subsets are locally Euclidean and

second countable. Finally,  $M_0 \sqcup M_1$  is also Hausdorff, because if  $(m, i), (n, j)$  are distinct elements of  $M_0 \sqcup M_1$ , then either  $i \neq j$ , in which case they can be separated by the disjoint open subsets  $\iota_i(M_i)$  and  $\iota_j(M_j)$ , or we have  $i = j$ , in which case there exist disjoint open subsets  $U, V \subseteq M_i$  with  $m \in U$  and  $n \in V$ , so that  $(m, i)$  and  $(n, i)$  are separated by  $\iota_i(U)$  and  $\iota_i(V)$ . In conclusion,  $(M_0 \sqcup M_1, \mathcal{T})$  is a topological manifold.

Let us now endow  $(M_0 \sqcup M_1, \mathcal{T})$  with a smooth structure. As  $\iota_i$  is an open injection, we can consider the following collection  $\iota_{i,*}(\mathcal{A}_i)$  of charts on  $M_0 \sqcup M_1$ :

$$\iota_{i,*}(\mathcal{A}_i) := \{(\iota_i(U), \varphi \circ \iota_i^{-1}) \mid (U, \varphi) \in \mathcal{A}_i\}, i \in \{0, 1\}.$$

It is straightforward to check that  $\iota_{0,*}(\mathcal{A}_0) \cup \iota_{1,*}(\mathcal{A}_1)$  is a smoothly compatible atlas on  $M_0 \sqcup M_1$ , and therefore induces a smooth structure  $\mathcal{A}$  on  $M_0 \sqcup M_1$  by *Proposition 1.8(a)*. As a final remark on this abstract construction, note that  $\iota_i$  is a diffeomorphism onto the open subset  $\iota_i(M_i)$  of  $M_0 \sqcup M_1$ . This essentially follows from the fact that for any smooth chart  $(U, \varphi)$  on  $M_i$  we have the smooth chart  $(\iota_i(U), \varphi \circ \iota_i^{-1})$  on  $M_0 \sqcup M_1$ .

With this construction at hand, it is straightforward to solve the exercise. Consider  $M_0 = \mathbb{R}$  and  $M_1 = \mathbb{R} \setminus \{0\}$ , both endowed with the standard smooth structure. Let  $M = M_0 \sqcup M_1$  be the smooth manifold which is their disjoint union. Consider the map

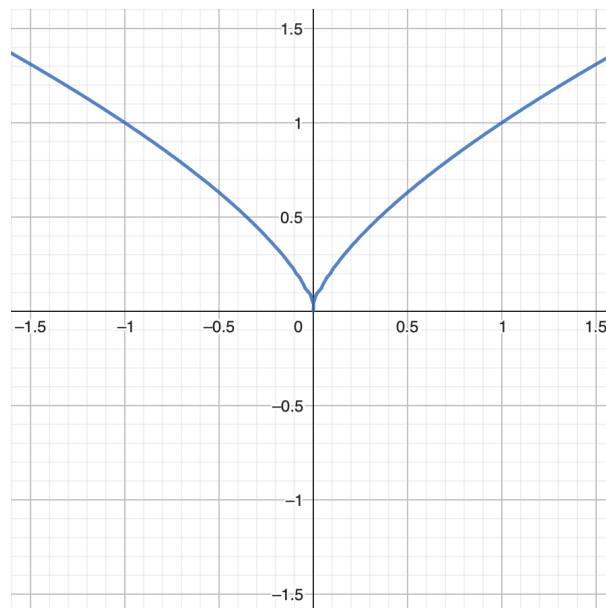
$$F: M \rightarrow \mathbb{R}^2 \\ (m, i) \mapsto (m, (-1)^i m).$$

Notice that  $F \circ \iota_0: \mathbb{R} \rightarrow \mathbb{R}^2$  is the map  $x \mapsto (x, x)$  and  $F \circ \iota_1: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^2$  is the map  $x \mapsto (x, -x)$ , and that both of these maps are smooth embeddings. As being continuous resp. smooth resp. a smooth immersion can be checked on an open cover, and  $\iota_i$  are diffeomorphisms, we conclude at once that  $F$  is continuous, smooth, and even a smooth immersion. Observe also that  $F$  is injective. Therefore, by *Proposition 5.16* the image  $\Phi^{-1}(0) = F(M)$  is an immersed submanifold of  $\mathbb{R}^2$ .

(b) The level set

$$\Psi^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^3 = 0\}$$

is plotted below.



We assume that  $\Psi^{-1}(0)$  can be given a topology and smooth structure making it into an immersed submanifold of  $\mathbb{R}^2$  and we will derive a contradiction using *Exercise 3(a)*. To this end, observe that  $\Psi^{-1}(0)$  must be 1-dimensional; indeed,  $\Psi^{-1}(0) \setminus \{(0, 0)\}$  is an embedded 1-submanifold of  $\mathbb{R}^2$ , as its two connected components, corresponding to  $(x, y) \in \Phi^{-1}(0)$  with  $x < 0$  (the left branch) and  $(x, y) \in \Phi^{-1}(0)$  with  $x > 0$  (the right branch), are the graphs of the smooth functions  $x \in (-\infty, 0) \mapsto x^{\frac{2}{3}}$  and  $x \in (0, +\infty) \mapsto x^{\frac{2}{3}}$ , respectively. Therefore,  $T_{(0,0)}\Psi^{-1}(0)$  is a 1-dimensional subspace of  $T_{(0,0)}\mathbb{R}^2 \cong \mathbb{R}^2$ , so by *Exercise 3(a)* there exists a smooth curve  $\gamma: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^2$  whose image lies in  $\Psi^{-1}(0)$  and which satisfies  $\gamma(0) = (0, 0)$  and  $\gamma'(0) \neq 0$ . Writing  $\gamma(t) = (x(t), y(t))$ , we see that  $y(t)$  takes a global minimum at  $t = 0$ , so  $y'(0) = 0$ . On the other hand, since  $\gamma(t) \in \Psi^{-1}(0)$  for every  $t \in (-\varepsilon, \varepsilon)$ , we have  $x^2(t) = y^3(t)$  for every  $t \in (-\varepsilon, \varepsilon)$ . Differentiating twice and setting  $t = 0$ , we obtain  $x'(0) = 0$ , and since  $y'(0) = 0$ , we conclude that  $\gamma'(0) = 0$ , which is a contradiction. Hence,  $\Psi^{-1}(0)$  is *not* an immersed submanifold of  $\mathbb{R}^2$ .

*Remark.* Here are a few remarks on the above solution to *Exercise 5(b)*.

- One could perform the construction of  $M_0 \sqcup M_1$  a bit more concretely: if  $M_0 \subseteq \mathbb{R}^n$  and  $M_1 \subseteq \mathbb{R}^n$  are given as embedded submanifolds of  $\mathbb{R}^n$ , then it is more or less straightforward to see that  $(M_0 \times \{0\}) \cup (M_1 \times \{1\}) \subseteq \mathbb{R}^{n+1}$  is an embedded submanifold of  $\mathbb{R}^{n+1}$ . The abstract disjoint union  $M_0 \sqcup M_1$  is then diffeomorphic to  $(M_0 \times \{0\}) \cup (M_1 \times \{1\})$  via the obvious map.

In our example where  $M_0 = \mathbb{R}$  and  $M_1 = \mathbb{R} \setminus \{0\}$ , this shows that  $M_0 \sqcup M_1$  is diffeomorphic to the embedded submanifold

$$\{(x, 0) \mid x \in \mathbb{R}\} \cup \{(x, 1) \mid x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2.$$

However, conceptually it is cleaner to argue just with the abstract disjoint union  $M_0 \sqcup M_1$ .

- The abstract disjoint union has a universal property, similar to the universal property of a product: for all smooth manifolds  $N$  and all smooth maps  $f_0: M_0 \rightarrow N$  and  $f_1: M_1 \rightarrow N$ , there exists a unique smooth map  $f: M_0 \sqcup M_1 \rightarrow N$  such that  $f \circ \iota_i = f_i$  for all  $i$ . In fact, for those who are familiar with the language of category theory, the disjoint union is the coproduct in the category of smooth manifolds.
- By *Proposition 5.16* there is a topology and smooth structure on  $F(M)$  making it an immersed submanifold of  $\mathbb{R}^2$ . The topology is given by

$$\mathcal{T} = \{F(U) \mid U \subseteq M \text{ open}\}.$$

This is a strictly finer topology on  $F(M)$  than the subspace topology. Indeed, by *Proposition 5.16*, the map  $(F(M), \mathcal{T}) \rightarrow \mathbb{R}^2$  is continuous, so  $\mathcal{T}$  contains the subspace topology. On the other hand, we have  $F(\iota_0(M_0)) \in \mathcal{T}$ , but

$$F(\iota_0(M_0)) = \{(x, x) \mid x \in \mathbb{R}\},$$

which is not open in the subspace topology on  $F(M)$  (otherwise it would have to contain  $F(M) \cap B(0, \varepsilon)$  for  $\varepsilon$  sufficiently small, which is certainly false).

- This exercise also demonstrates that the topology and smooth structure on an immersed manifold might not be unique (cf. *Exercise 2*). Indeed, the map

$$\begin{aligned} G: M &\rightarrow \mathbb{R}^2 \\ (m, i) &\mapsto (m, (-1)^{i+1}m) \end{aligned}$$

is also an injective smooth immersion with image  $\Phi^{-1}(0)$ , and therefore induces a topology and smooth structure on  $\Phi^{-1}(0)$  making it an immersed submanifold of  $\mathbb{R}^2$  by *Proposition 5.16*. But one can check that it is different from the one induced by  $F$ .